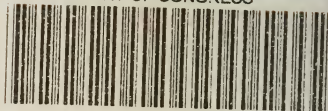


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NOTES

ON

ASTRONOMY:

COMPILED FROM THE LECTURES

OF

*Charles William*  
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# ASTRONOMY.

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## PART FIRST. QUESTIONS ON ASTRONOMY.

### 1. *Into what classes is Astronomy divided?*

(1.) Practical Astronomy, which includes the use of instruments, and the application of the mathematical principles, necessary for the determination of the places of the heavenly bodies, or the co-ordinates of their positions, together with the necessary corrections of these co-ordinates.

(2.) Theoretic Astronomy, or the theory of the motions of the heavenly bodies, and the nature and position of their orbits, derived from the results of practical astronomy, by the aid of the theory of conic sections.

(3.) Descriptive Astronomy, which comprehends the physical constitution and phenomena presented by the heavenly bodies, individually. This branch requires the aid of a powerful telescope: the most distinguished names that occur in its history and development are the two Herschels, and Schroeder, of Lilienthal.

(4.) Physical Astronomy, which seeks to investigate all the movements of the material universe, as a purely mechanical problem. It is also known under the name of "Celestial Mechanics," and has been illustrated by La Place and La Grange. This branch involves the highest mathematics.

### 2. *What are the proofs of the figure of the Earth?*

(1.) It has been circumnavigated.

(2.) As a traveler proceeds directly north or south, the same stars cross the meridian higher or lower in the diurnal

motion, the change being gradual as the traveler advances ; this indicates a gradual curvature of the earth in this direction.

(3.) The circular form of the horizon at sea. If the earth were a sphere, and the visual ray drawn from the eye of the observer, tangent to the surface of the earth, were set in rotation about the diameter of the earth, passing through the eye as an axis, it (the visual ray) would describe a visual cone, whose circle of contact with the sphere would be the boundary of the visual horizon. The identity of the figure with that actually observed is a proof of the spheroidal figure of the earth.

(4.) The form of the earth shown in eclipses of the moon.

(5.) The circumstances attending the arrival of ships from sea. At first, only the tops of the masts are observed, then we see the yards, one after another, until at last, as they come nearer, we discern the whole fabric.

(6.) The analogy of form, between all the other bodies, composing the solar system.

(7.) The exact spheroidal figure of the earth is ascertained by minutely accurate measurements in the direction of meridians and parallels, together with astronomical observations. See *Dr. Hackley's Trigonometry*, p. 366.

### 3. *What are the proofs of the Earth's Rotation?*

(1.) The apparent diurnal rotation of the sun, moon, planets, and fixed stars, at such immensely varying distances from the earth, make the probabilities a thousand to one that it is the earth that is in rotation on its axis, rather than that such an immense number of objects, so vast and so remote, should have their motions of translations so exactly adjusted in proportion to their distance from the earth, as to conspire in producing the effects we see.

(2.) When a body is dropped from the top of a high tower, it will not fall exactly at the base of the tower, or in a perpendicular to the surface of the earth from the point whence it was allowed to fall, but it will fall a little east of the perpendicular, the velocity communicated at that greater height carrying the body further on, than if it had remained at the surface of the earth.

(3.) The pendulum experiment. By the property of inertia, the tendency of matter is to continue a motion, which it has received by the application of some force, in a constant direction, unless some extraneous force change it. Thus, a pendulum, set in motion in a certain plane, would ever oscillate in that plane, if the point of suspension be perfectly free to revolve on a pivot; or if, instead of a rod, a thread without torsion be used, its pendulum will not feel the rotation of the point of suspension, or of the surface of the earth under it, with which it is connected in the diurnal motion. Now, if a circle be described about the horizontal projection of the point of suspension, in a manner like the dial of a clock, the points of the dial over which the pendulum oscillates will be found to slowly change, and to complete the whole circumference (the amount depends on the latitude) in about twenty-four hours.

4. *How is an orbit given?*

By its elements.

5. *How many elements of an orbit are there?*

Seven.

6. *Which two determine the position of the plane of the orbit in space?*

The longitude of the node and the inclination.

7. *Which two elements determine the size and shape of an orbit in its own plane?*

The semi-major axis (called also the mean distance) and the eccentricity.

8. *What element determines the position of an orbit in space?*

The longitude of the perihelion or vertex of the ellipse nearest the focus in which the sun is placed.

9. *What two determine the position of the planet in its orbit at any given time?*

The epoch of the perihelion passage and the periodic time.

10. *Which two of the seven elements do not occur in the orbit of the earth?*

The longitude of the node and the inclination.

11. *What is the node of a planet's orbit?*

It is the point in which the orbit of the planet intersects the orbit of the earth, *i. e.*, the plane of the ecliptic.

12. *What is that node called, through which a planet passes in going from the south to the north side of the ecliptic?*

The ascending node. ( $\Omega$ )

13. *What is that called, passed through in the passage from north to south?*

The descending node ( $\Upsilon$ .)

14. *What is the line of nodes?*

It is a line joining the nodes of a planet's orbit, and which always passes through the sun, or it is the intersection of the plane of the planet's orbit with the plane of the ecliptic.

15. *What is the longitude of the node?*

The angle which the line of nodes makes with the line of equinoxes. (This angle being given, and a point on the plane of the ecliptic, *viz.*: the position of the sun, through which it passes, the position of this line will be entirely determined; and then the angle, which the plane of the orbit passing through this line makes with the plane of the ecliptic, entirely determines the position of the plane of the orbit in space.)

16. *What is the line of equinoxes?*

It is a line drawn from the sun to the equinoctial point. (A line drawn from the earth would be sensibly parallel to this, since the equinoctial point may be considered as at an infinite distance in comparison with the distance of the earth from the sun. The latter line is also the line of intersection between the plane of the equator and the plane of the ecliptic.)

17. *What is the longitude of the perihelion?*

It is the angle which the projection of the major axis of the orbit, on the plane of the ecliptic, makes with the line of equinoxes. (This angle being given, the position of this projection, since it must pass through the center of the sun, is known on the plane of the ecliptic. A plane, through this projection, perpendicular to the plane of the ecliptic, will intersect the plane of the orbit, given in position by the first two elements as the major axis of the orbit; the posi-

tion of which, and therefore the orbit in its own plane, thus becomes known.)

18. *What is the epoch of the perihelion passage?*

It is the instant at which the planet passes the perihelion of its orbit. (The perihelion is the point of the orbit nearest the sun, or to that vertex of the ellipse next to that focus on which the sun is placed.)

19. *What is the radius vector of a planet?*

It is the distance from the center of the sun to the point in the orbit which the planet has reached.

20. *What is the "anomaly?"*

It is the angle which the radius vector makes with the major axis; or, it is the angular distance of the planet from the perihelion. This is known as the "true anomaly."

21. *What is the "mean anomaly?"*

It is the angular distance from the perihelion which a planet would have, at any given time, if it moved with an uniform velocity, and completed its *period* in the same time that it does with a varying velocity.

22. *To what is the true angular velocity of a planet in its orbit proportional?*

It is inversely as the *square of the radius vector*; i. e.: the square of the distance from the sun. It is, therefore, greatest when the planet is at the perihelion, and least when the planet is at the aphelion, or other vertex of the ellipse.

23. *How, from the sixth and seventh elements in a planet's orbit, is the mean place of the planet found?*

The "mean anomaly" will be that portion of three hundred and sixty degrees which the time elapsed, at the instant for which the mean position is required, since the epoch of the perihelion passage, is of the whole periodic time.

24. *How is the true place of the planet in its orbit, or the "true anomaly" found from the "mean?"*

By Kepler's problem, which gives the means of determining a third "anomaly," called the "eccentric," from the "mean," and the means of determining the "true" from this "eccentric."

25. *What is the heliocentric latitude of a heavenly body?*

It is the angle which a line drawn from the center of the sun to the body, (the radius vector in case of a planet,) makes with the plane of the ecliptic.

26. *What is the geocentric latitude of a heavenly body?*

It is the angle which a line drawn from the center of the earth to the body makes with the ecliptic.

27. *What is the heliocentric longitude of a heavenly body?*

It is the angle which the projection of the line drawn from the center of the sun to the heavenly body, (on the plane of the ecliptic,) makes with the line drawn from the center of the sun to the vernal equinox.

28. *What is the geocentric longitude?*

It is the angle made by the projection of a line drawn from the center of the earth to the heavenly body, (on the plane of the ecliptic,) with the line of equinoxes.

29. *What is the apparent diameter of a heavenly body?*

It is the angle subtended at the eye of the observer by the diameter of the disc of the body.

30. *What is the line of apsides?*

The line joining the perihelion and aphelion of an orbit, or it is the major axis of the orbit.

31. *What are the apsides of an orbit?*

The perihelion and aphelion.

32. *What is called the lower apsis?*

The perihelion.

33. *What the upper apsis?*

The aphelion.

34. *What is the equation of the center?*

It is the difference between the "true" and "mean" anomalies, *i. e.*, the difference between the true and mean radii vectores.

35. *How is the "mean anomaly" obtained from tables of the sun?*

By taking the difference between the mean longitude of the sun and the mean longitude of the perigee.

36. *What is a sidereal year?*

The interval between the instants at which the sun has the same difference of right-ascension from any star.

37. *How long is this interval ?*

At the mean, 365.256 336 days.

38. *How does this compare with the tropical year ?*

It is somewhat longer.

39. *To what is this owing ?*

The precession of the equinoxes.

40. *What is the physical cause of precession ?*

The spheroidal figure of the earth, (produced by a protuberant mass at the equator,) acted upon by the combined attraction of the sun and moon.

41. *What is the effect of this attraction ?*

To produce a slow revolution of the axis of the equator about the axis of the ecliptic.

42. *What is the expression for the period of revolution ?*

$360^{\circ}$

$50''2 =$  in round numbers, 26,000 years.

43. *What produces the precession of the equinoxes ?*

The revolution of the axis of the equator, carrying the equator with it, and retaining its plane always perpendicular.

44. *Into how many classes are the disturbances of elements divided ?*

Two.

45. *What are they ?*

Secular and periodic.

46. *What difference is there between them ?*

The periods in which the secular complete themselves are very long ; those of the periodic much shorter.

47. *What must be done, in order to have the exact value of any element, as the obliquity of the ecliptic, for instance, at any given time ?*

The combined effect of each of these two classes must be computed.

48. *Where would you find the mean obliquity, obtained in this way, at the beginning of the year ?*

In the Nautical Almanac, just before the ephemerides of the planets.

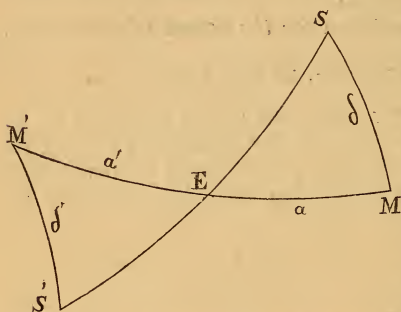
49. *In what does the actual obliquity differ from the mean ?*

By the effect of the <sup>m</sup>utation.

## PART SECOND. PROBLEMS.

PROBLEM I.—*To determine the instant of the Equinox, and the exact position of the Equinoctial points.*

Let the declination of the sun be observed daily with the mural circle, and the difference between the right ascension of the sun, and some star, with the transit instrument, as the sun crosses the meridian, each day for some days, about the time of the equinox.



Suppose  $MM'$  to be the arc of the equator comprehended between the declination circles  $MS$  and  $M'S'$ , passing through the sun at two noons near the equinox, the one before, the other after. The declination of the sun having changed from north to

south of the equator, or *vice versa*, in the interval, as, from  $S$  to  $S'$ , or from  $S'$  to  $S$ ,  $MM'$  will be the difference of the right-ascensions of the sun, at the two observations; each right-ascension of the sun having been estimated from the declination circle passing through the star, say  $\alpha$  Lyrae as a zero.  $MS$  and  $M'S'$  will be the observed declinations of the sun, represented by  $\delta$  and  $\delta'$ : the corresponding right-ascensions, estimated from the equinoctial point  $E$ , in opposite directions, being  $a$  and  $a'$ . By the solution of the right-angled spherical triangles,  $EMS$  and  $EM'S'$ , (*Trigonometry*, p. 143.)

$\cot E = \frac{\sin a}{\tan \delta}$  and  $\cot E = \frac{\sin a'}{\tan \delta'}$ ; whence  $\frac{\sin a}{\sin a'} = \frac{\tan \delta}{\tan \delta'}$  by equating the second members; *i. e.*,  $\sin a : \sin a' :: \tan \delta : \tan \delta'$ , or, by composition and division,  $\sin a + \sin a' : \sin a - \sin a' :: \tan \delta + \tan \delta' : \tan \delta - \tan \delta'$ , or (*Trigonometry*,

p. 78)  $\tan \delta + \tan \delta' : \tan \delta - \tan \delta' :: \tan \frac{1}{2} (a + a') : \tan \frac{1}{2} (a - a')$ . The last term of this proportion, being the only one unknown, may be computed: then  $\frac{1}{2} (a + a')$  and  $\frac{1}{2} (a - a')$  being known, by taking their sum and difference,  $a$  and  $a'$  will be obtained. The difference of right-ascension between the sun and equinoctial point, at either of the two noons, being now known, and the difference of right-ascension between the sun, at this noon, and  $\alpha$  Lyræ, being also known by observation, the difference of right-ascension between  $\alpha$  Lyræ and the equinox, becomes known. Such observations should be repeated for several days before and after the equinox, and the mean of all the results be taken as the true difference of right-ascension between  $\alpha$  Lyræ and the equinox; in other words, the true right-ascension of  $\alpha$  Lyræ, estimated from the equinox as the zero. If now the clock be set to indicate the hours, minutes, and seconds, expressed by the right-ascension of  $\alpha$  Lyræ in time, found, as before, at the instant that the star passes the meridian, the clock will indicate no hours, no minutes, no seconds (or the zero of time), when the vernal equinox comes to the meridian, and will indicate the right-ascension of any other heavenly body, in time, when that body comes to the meridian.

#### PROBLEM II.—*Precession of the Equinoxes.*

If the autumnal equinox be observed in the same way as the vernal, its place will be found to differ 12 hours in right ascension, or exactly  $180^\circ$  from the vernal equinox; *i. e.*, they are diametrically opposite to each other on a line passing through the centre of the earth, and extending to the celestial sphere. It will be found, in computing the difference of right-ascension between the equinox and some star, like  $\alpha$  Lyræ, as above, that the equinoctial point is not fixed, but that its difference of right-ascension from any star is all the while increasing, at a rate, if we take a mean of many years, of about  $50''.2$  in arc annually; this is called "the precession of the equinoxes." The late Mr. Bessel, of Königsberg, gives the exact expression for the mean precession of one year, by the formula  $50''.21129 + 0.0002443t$ ;

and therefore it is impossible

in which  $t$  denotes the number of years elapsed, at the time required, since 1750, A. D.

### PROBLEM III.—*The Calendar.*

The mean of a great number of intervals between two successive equinoxes, in time, is 365.242217 days, or 365d. 5h. 48m. 58s. This is called a tropical year, and its exact length, as thus determined, is of importance in the regulation of the calendar, by which the labors of the husbandman are governed. If the year be supposed to consist of exactly 365 days, as was the case before the Julian Calendar, then at the end of 4 years, the fraction by which the true tropical year exceeds 365 days, amounts to nearly a day, which the sun has to move in declination, before its declination attains the same value it had attained on the same day of the month 4 years before. The improved Calendar, made by the decree of Julius Cæsar, required the intercalation of one day in 4 years, by making 29, instead of 28 days in February, so that on the first of March, the sun would have attained the same declination that it had on the first of March 4 years anterior. But the Julian Calendar itself required a correction in the lapse of centuries, owing to the error in taking the fraction over 365 days, in the length of the true tropical year; to be exactly 6 hours. This error was, however, the other way: 365 days were too little; 365d. 6h. too much: the fraction by which it was too much accumulated so as to amount to 3 days in 400 years. Instead, therefore, of intercalating a day at the end of February, we omit to intercalate once in 100 years for 3 successive centuries, as, for instance, in the years 1700, 1800, 1900; but in the fourth century, as 2000, intercalate, so as to omit three intercalations in 400 years, it requires 3500 years for the error of this mode of correction to amount to a day; the correction being, truly, a little over 3 days in 400 years. This last reformation in the Calendar, is called the Gregorian, from Pope Gregory XIII., who made it in 1582. It is immaterial, for the interests of agriculture, by what name any day in the year is called: it is only important that the day on which the sun attains a certain declination, north or

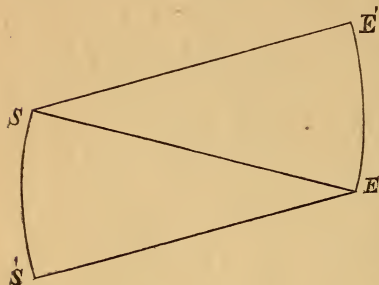
south, should be called by the same name every year. Gregory chose to go back to the Council of Nice, A. D. 325, in which year the equinox happened on the 21st of March, and so to arrange as to have it happen on the 21st of March ever after. As the error in the interval from 325 to 1582, some 1300 years, at the rate of 3 days in every 400 years, amounted to 10 days, Gregory ordered the 5th of October, 1582, to be called the 15th. This would make the vernal equinox happen on the 21st of March, 1583, as it did in 325; otherwise, it would have happened on the 11th of March, 1583. This omission of 10 days all at once, made the difference between what is called the old and new style. The new style was not adopted at once in Protestant countries; in England, not till 1750, when 11 days were taken out, the 3d of September being called the 14th, owing to the intercalary day to be omitted in the year 1700. There is now a difference of 12 days between old and new styles, owing to the intercalary day that was omitted in 1800.

PROBLEM IV.—*Determination of the Obliquity of the Ecliptic.*

The position of the equinoxes being accurately determined by a method already given, a single observation of the right ascension and declination of the sun, will serve to determine the obliquity of the ecliptic (*Trigonometry*, p. 143); the mean of a great many results should be taken. If the obliquity be computed from observations taken at very long intervals, it will be found to have undergone a change, of which physical astronomy teaches us the cause; it results from the disturbing action of the other planets of the solar system. The obliquity is now diminishing; after a very long period, it will begin to increase again; and so will it continue to move back and forth for ever; in the language of Pontecoulant, "pendule immense qui batte des siecles"—"an immense pendulum whose oscillations are continuous." The annual diminution of the obliquity, in the present age, is  $0''.457$ .

*Centuries*

PROBLEM V.—*To prove that the apparent orbit of the sun around the earth is equal, in all respects, to the real orbit of the earth around the sun.*



Let  $EE'$  be a portion of the real orbit of the earth, whatever be the curve, around the sun at  $S$ . When the earth is at  $E$  the sun is seen in the direction  $ES$ . Suppose, now, the earth to have moved to  $E'$ , the sun will be seen in the direction  $E'S$ , the same direction

as  $ES'$  parallel to  $E'S$ , *i. e.*, the same direction as if the earth had continued stationary at  $E$ , and the sun had moved from  $S$  to  $S'$ . The two sectors,  $ES E'$  and  $S E S'$ , having their angles and radii equal, are equal; their angles being angles which parallel lines make with  $ES$ , and their radii being the distance between the earth and sun. Since the same may be proved in the same way of all the elemental sectors of which the terrestrial and apparent solar orbits are respectively composed, it follows that the whole orbit of the earth around the sun, which is made up of the former, is equal to the whole apparent orbit of the sun around the earth, which is made up of the latter.

The determination of the size and shape of the one orbit, then, will be the determination of the size and shape of the other.

PROBLEM VI.—*The first approximation of the apparent solar orbit may be obtained as follows :*

Let the right ascensions and declinations of the sun be observed with the transit instrument and mural circle, and let the apparent diameter of the sun at the same time be observed with the micrometer, every few days throughout the year. Let the observed right ascensions and declinations be converted into longitude by the method pointed

out in *Trigonometry*, p. 143. Draw lines radiating from a point and making, with each other, angles equal to the difference of longitude, found as above, throughout the year, and lay off on these lines distances proportional to the observed diameter of the sun at the same time (which observed apparent diameters will be inversely proportional to the distances of the sun from the earth); a curve traced through the points thus determined on these lines, will give the approximate form of the orbit, which will be found to be nearly an ellipse.

PROBLEM VII.—*Continuation of former.*

Instead of this graphic construction, the radii-vectores of an ellipse, the pole being at the focus, with the semi-major axis unity, and the eccentricity equal to the ratio of the sum and difference of the greatest and least apparent diameters of the sun, may be computed for variable angles, equal to the difference of the sun's longitude when his semi-diameter is greatest; and the various other values of the longi-

tude by the formula  $r = \frac{a(1-e^2)}{1+e \cos v}$ , or, rather, since  $a$  is

unity,  $r = \frac{1-e^2}{1+e \cos v}$ ; and the length of these radii-vectores,

thus computed, will be found to bear the same proportion to unity that the corresponding apparent diameters do to half the sum of the greatest and least apparent diameters, thus identifying the orbit with the ellipse. If the areas of the sectors, comprehended between the radii-vectores, be computed, they will be found proportional always to the time employed by the radius-vector in describing them.

The above investigation into the nature of the earth's orbit has developed two of the three celebrated laws of Kepler, which apply to the orbits of all the planets.

PROBLEM VIII.—*Kepler's Laws.*

1. The orbits are all ellipses, of which the sun is one of the foci.

2. The areas described by the radius-vector of a planet around the sun are proportioned to the times employed in describing them.

3. The squares of the periodic times of the planets are as the cubes of their mean distances from the sun, or the cubes of the semi-major axes of their orbits. That the semi-major axis is the mean of all the distances from the different points of the periphery of an ellipse to the focus appears, not only from its being half the sum of the greatest and least distances, but also from a well-known property of the ellipse, that the sum of the distances of any point from the two foci is constant and equal to the major axis. Owing to the symmetrical form of the ellipse, there will always be two points at the same distance from one focus that one of them is from the two foci. The points of the curve may, therefore, be arranged in pairs, the sum of the radii-vectores in every pair being equal to the major axis, and the mean of the whole, therefore, will be equal to half this line.

PROBLEM IX.—*Relation between the Radius Vector and the Velocity.*

As the areas described in equal times by the radius vector, are equal (see Kepler's Law), the angular velocity of the radius vector at the perihelion must be greater, in order that the width of the sector described at that part of the ellipse in a given time, may be sufficiently greater to make up for its diminished length. There is, in fact, an exact relation between the angular velocity and the length of the radius-vector at all points of the orbit; the former being inversely proportional to the square of the latter; for, if  $v$  denote the angle described by the radius vector in a unit of time (as the distance passed over in a unit of time, is the definition of velocity, so the angle described in a unit of time, is the angular velocity), or the arc of a circle whose radius is unity which measures that angle, then  $r v$  would express the similar arc whose radius is  $r$ , since similar arcs, or arcs corresponding to equal angles in different circles, are to each other as their radii. If this  $r$  be the radius vector,

and the small arc of the elliptical orbit described in a unit of time, say one day, be considered the arc of a circle,—which it may be, it is so small—the area of a corresponding sector will be obtained by multiplying this arc  $r v$  by  $\frac{1}{2}r$ , so that we have the area of a sector described in a unit of time equal  $\frac{1}{2}r^2 v$ . Denoting this area described in a unit of time, by  $a$ , which, according to the second law of Kepler, is the same or constant in every part of the ellipse, we have

$a = \frac{1}{2}r^2 v$ , whence  $v = \frac{2a}{r^2}$ ; i. e., the velocity varies inversely

as the square of the radius vector, the numerator in the value of  $v$ , being constant; or, clearing the last equation of fractions, we have  $vr^2 = 2a$ , and if  $v'$  and  $v''$  denote two particular values of  $v$ , and  $r'$  and  $r''$  the corresponding values of  $r$ , we have  $v'r'^2 = v''r''^2$ , whence  $\frac{v'}{v''} = \frac{r''^2}{r'^2}$  or,  $v' : v'' :: r''^2 : r'^2$

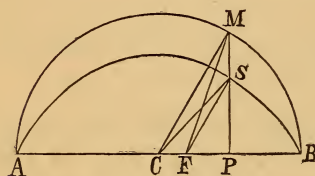
**PROBLEM X.**—*Longitude of the perihelion of the earth's orbit, or longitude of the perigee of the solar orbit.*

The orbit of the earth, or apparent solar orbit, being symmetrical to the line of apsides, or major axis, the two registered longitudes of the sun which differ  $180^\circ$ , and the times of observations for which differ by half the whole periodic time, will be the longitudes of the perihelion and aphelion points, or apsides of the orbit. No other straight line, than the line of apsides, drawn through the focus, divides the area of the ellipse into two equal parts.

*Note.*—If the longitude of the perihelion and aphelion be determined, as above, at distant epochs, it will be found to have undergone a change in the interval, greater than would have been occasioned by the precession of the equinoxes. The annual increase of the longitude of the perigee, or point of the apparent solar orbit nearest the earth, is at a mean  $61''.9$ , and, as the retrograde motion of the equinoxes is  $50''.2$  along these elliptic from east to west, there remains  $11''.7$  of actual progression in space, made by the perigee from west to east, annually. The anomalistic year is the time occupied by the sun in passing from the perigee

to the perigee again, or the interval between two consecutive times when the sun has the same anomaly. It exceeds the sidereal year, by the time which the sun would occupy in moving the  $11''.7$  above mentioned, in longitude; and it exceeds the tropical year by the time occupied by the sun in moving  $61''.9$ . The determination of the mean anomalistic year, does not require a very accurate determination of the epoch, or longitude of the perigee. By taking the epoch of the perigee at very distant intervals of time, and dividing the whole interval expressed in days and fractions of a day, by the number of years intervening between the two epochs, the quotient will be the length of the mean anomalistic year in days and fractions of a day.

PROBLEM XI.—*Kepler's Problem to determine the true anomaly from the mean by means of the eccentric.*



In the diagram, A M B is a semi-circle on the major axis A B as a diameter, the earth being at the focus E, and the sun at S in the apparent solar orbit; the angle B E S is the true, and B C M the eccentric

anomaly.

In this problem we are to find the relation between the mean and eccentric anomaly.

If the semi-axes of the orbit be denoted by  $a$  and  $b$ , since the ordinate of a circle : the ordinate of the ellipse ::  $a$  :  $b$  (see *Analytical Geometry*), and since all the ordinates of the ellipse constitute the area of the circle, and since, by the theory of proportion, the sum of the antecedents : the sum of the consequents :: any one antecedent : its consequent, we have the area of the circle : the area of the ellipse ::  $a$  :  $b$ , and, for a similar reason, the area of the segment B M P : the area of the segment B S P ::  $a$  :  $b$ , a similar ratio. The triangles E P M and E P S have also the same ratio, of  $a$  to  $b$ , since they have the same base E P, and are, therefore, to each other as their altitudes. The sum of the triangles and segments, or the sectors B E M and B E S, have, therefore,

*equal to all the ordinates of the circle  
the area of the*

the same ratio. If  $T$  denote the periodic time, or time of describing the whole ellipse, and  $t$  that of describing the sector  $BES$ , and  $a=1$ , let the area of the circle be equal to  $\pi$ ; and, since the areas are proportional to the times, the area of the sector  $BEM$  will be expressed by the fraction

$\pi \frac{t}{T}$ . The area of the triangle  $MCE$  (if  $e$  denote the base  $CE$ , and  $u$  denote the eccentric anomaly, *i. e.*, the angle  $C$ , or arc  $MB$ ) is  $\frac{1}{2}e \sin u$ . Adding the expressions for the areas of the sector and triangle, we have the area of the sector  $BCM$  (which is also expressed, since  $a=1$ , by  $\frac{1}{2}u$ ) equal to  $\pi \frac{t}{T} + \frac{1}{2}e \sin u = \frac{1}{2}u$ , whence  $2\pi \frac{t}{T} = u - e \sin u$ ; but

$2\pi \frac{t}{T}$  expresses the angular motion of  $S$  about  $E$ , in a unit of time, say one day; *i. e.*, the mean daily motion, if  $T$  be expressed in days, because  $2\pi$  is the circumference whose radius is unity, and  $T$  the number of days in a complete revolution. Denoting  $\frac{2\pi}{T}$  by  $n$ ,  $nt$  will evidently be the mean anomaly, or mean angular motion, in the time or number of days expressed by  $t$ : the last equation thus becomes  $nt = u - e \sin u$  (1.) or the mean anomaly in terms of the eccentric,  $nt$  being the mean anomaly, and  $u$  the eccentric.

PROBLEM XII.—*To find the relation between the eccentric anomaly and the true anomaly.*

We have (by analytical geometry) the expression for the distance between two points in space.

$$(x' - x'')^2 + (y' - y'')^2 \quad (K)$$

If the two points be  $E$  and  $S$ , (see last diagram,) denoting the distance by  $r$ , namely,  $ES$ :  $x'$  of the above equation in this particular case takes the value of  $e$ ,  $y' = 0$ ;  $x'' = \cos u$ ;

$y'' = \frac{b}{a} \sin u = (\sqrt{1 - e^2}) \sin u$ ; and  $(K)$  becomes

$$r^2 = (e - \cos u)^2 \pm (1 - e^2) \sin^2 u = e^2 - 2e \cos u + \cos^2 u + \sin^2 u - e^2 \sin^2 u.$$

2\*

*and since BES bears the same proportion to the area of the ellipse that BEM does to the area of the circle -*

But  $\sin^2 u + \cos^2 u = 1$ ; therefore

$$r^2 = 1 + e^2 - 2e \cos u - e^2 \sin^2 u, \text{ or}$$

$$1 + e^2 - 2e \cos u - e^2 (1 - \cos^2 u) = 1 - 2e \cos u + \cos^2 u e^2.$$

$$\text{Therefore } r = 1 - e \cos u.$$

The polar equation of the ellipse, the origin of polar co-ordinates being at the focus, is  $\frac{1-e^2}{1+e \cos v}$ , in which  $v$  is

the true anomaly. Equating this value of  $r$  with that in the last equation, we have

$$1 - e \cos u = \frac{1-e^2}{1+e \cos v} \text{ therefore } 1 + e \cos v = \frac{1-e^2}{1-e \cos u} \text{ or,}$$

$$e \cos v = \frac{1-e^2}{1-e \cos u} - 1 = \frac{1-e^2-1+e \cos u}{1-e \cos u} = \frac{e \cos u - e^2}{1-e \cos u}.$$

$$\text{Dividing both sides by } e, \text{ we have } \cos v = \frac{\cos u - e}{1 - e \cos u}.$$

$$\text{By Trig., (page 100,) } \tan^2 \frac{1}{2} v = \frac{1 - \cos v}{1 + \cos v} =, \text{ by last equation,}$$

$$\frac{1 - \frac{\cos u - e}{1 - e \cos u}}{1 + \frac{\cos u - e}{1 - e \cos u}} = \frac{1 - e \cos u - \cos u + e}{1 - e \cos u + \cos u - e} =$$

$$\frac{(1+e)(1-\cos u)}{(1-e)(1+\cos u)} = \frac{1+e}{1-e} \tan^2 \frac{1}{2} u. \quad \text{Extracting square}$$

$$\text{root, we have } \tan \frac{1}{2} v = \sqrt{\frac{1+e}{1-e}} \times \tan \frac{1}{2} u \text{ (2), which}$$

expresses the relation between the true and the eccentric anomaly in each part. (In order to have convenient expressions for computation, it is necessary to develop (1) and (2) in series, for which see Bowdich's "La Place," page 374, vol. i. Convenient practical methods are given in the "Theoria Motus Corporum Cœlestium," of Gausse, pages 10 and 13.)

## PROBLEM XIII.

What is termed the angle of eccentricity, is the angle whose sine is the eccentricity. If we denote this angle by  $\phi$ , we have by definition,  $e = \sin \phi$  (1.) By *Trig.*, page 104, (18,) we have  $\sqrt{1-e^2} = \cos \phi$  (2) or  $(\sqrt{1+e})(\sqrt{1-e}) = \cos \phi$  (3). From (1) we have  $1-e = 1 - \sin \phi$  (4).

Dividing (4) by (3)  $\frac{\sqrt{1-e}}{\sqrt{1+e}} = \frac{1-\sin \phi}{\cos \phi} = \tan (45^\circ - \frac{1}{2} \phi)$ .  
(See *Trig.*, page 102 (45). Then from (2,) last problem,

$$\frac{\tan \frac{1}{2} u}{\tan (45^\circ - \frac{1}{2} \phi)} = \tan \frac{1}{2} v.$$

Given  $u = 324^\circ 16' 29'' 5$ ,  $\phi = 14^\circ 12' 1'' 87$ .

$$\frac{1}{2} u = 162^\circ 8' 14'' 75 \text{ Log. Tan.} = 9.5082198.$$

$$45^\circ - \frac{1}{2} \phi = 37^\circ 53' 59'' 07 \text{ Log. Tan.} = 9.8912427.$$

$$157^\circ 30' 41'' 5 = \text{Log. Tan. } \frac{1}{2} v = 9.6169771.$$

$$\text{Therefore } v = 315^\circ 1' 23''.$$

To compute the corresponding length of the radius vector from the ellipse, which is the orbit.

The polar equation is  $r = a \frac{(1-e^2)}{(1+e \cos v)}$  If we denote the

numerator (called the parameter) by  $p$ : then  $r = \frac{p}{1+e \cos v}$ , and then as  $1-e^2 = \cos^2 \phi$ , by (2) above, therefore  $p = a \cos^2 \phi$ .

*To compute now.*

$$\text{Given } \phi = 14^\circ 12' 1'' 87, \text{ Log. Sine} = \text{Log. } e = 9.3897262$$

$$v = 315^\circ 1' 25'' \text{ Log. cosine} = 9.8496597$$

$$e \cos v = 0.173545 = \text{Log. } e \cos v = 9.2393859$$

$$\text{Log. } 1+e \cos v = 0.0694959$$

$$\text{Given Log. } a = 0.4224389$$

$$\text{Log. cosine}^2 \phi = 9.9730448$$

$$\text{Log. } p = 0.3954837$$

$$\text{Log. } 1+e \cos v = 0.0694959$$

$$\text{Log. } r = 0.3259878$$

PROBLEM XIV.—*Gausse's Method of determining the eccentric anomaly from the mean.*

From (1) Prob. XI. we have  $u = nt + e \sin u$ . This equation being transcendental, does not admit of direct solution; it may be resolved by tentative methods, assuming an approximate value for  $u$ , and repeatedly correcting it until the equation shall be exactly satisfied. This, in practice, is much more convenient and expeditious than by a series as given by La Place.

Let us suppose  $\varepsilon$  to be the approximate value of  $u$  and  $\chi$  expressed in seconds to be the value of the correction to be added, in order to have the true value of  $u$ . So then  $u = \varepsilon + \chi$ . Let now the quantity  $e \sin \varepsilon$  be computed in seconds, by logarithms, noting the difference of the logs, for one second in that part of the tables where we have  $\sin \varepsilon$ , and the difference of logs, corresponding to a difference of unity in the numbers in that part of the table, where we have the number  $e \sin \varepsilon$ , and its log. Let these differences be denoted respectively by  $\lambda$  and  $\mu$ . If now we suppose the value of  $\varepsilon$  to be assumed so near the value of  $u$ , that we may suppose the difference of the logs to be proportional to the difference of the numbers, it is evident that we may place  $e \sin (\varepsilon + \chi) = e \sin \varepsilon \pm \frac{\lambda \chi}{\mu}$  (the sign being  $+$  in first and fourth quadrant, and  $-$  in 2d and 3d, because in 1st and 4th the sine is an increasing function of the arc.) *Therefore,*

Since  $\varepsilon + \chi = nt + e \sin (\varepsilon + \chi)$  or by last equation

$$(O) \quad \varepsilon + \chi = nt + e \sin \varepsilon \pm \frac{\lambda \chi}{\mu} \text{ by transposing}$$

$$\chi \mp \frac{\lambda \chi}{\mu} = nt + e \sin \varepsilon - \varepsilon, \text{ whence}$$

$$\chi = \frac{\mu}{\mu \mp \lambda} (nt + e \sin \varepsilon - \varepsilon) \text{ therefore by (O) above}$$

$$\varepsilon + \chi, \text{ or } u, = nt + e \sin \varepsilon \pm \frac{\lambda}{\mu \mp \lambda} (nt + e \sin \varepsilon - \varepsilon.)$$

PROBLEM XV—*Example, to obtain the eccentric from the mean anomaly.*

Given, the mean anomaly =  $332^{\circ} 28' 54.77''$ .

$\phi = 14^{\circ} 12' 1'' 87$ . Log. Sine = 9.3897262  
 Radius of circle =  $206264'' 806$ . Log. = 5.3144251

Eccentricity =  $e = 14^{\circ} 3' 20''$ . Log. Sine = 4.7041513

For a first calculation, let us assume  $\epsilon = 326^{\circ}$ , as the approximate value of the eccentric anomaly, which must be evidently less than the mean,  $360^{\circ} - 326^{\circ} = 34^{\circ}$ , and the log. sine of this is the same as the log. sine of  $\epsilon$ .

Log. Sin  $\epsilon = 9.74756$  negative.

Log. Sin  $34^{\circ} 1' = 9.74775$

Difference of logs.  $1' = 19$

Dividing by 60 we get the difference of logs. for  $1'' = 0.32$  as the value of  $\lambda$ .

Log. Sin  $\epsilon = 9.74756$  negative.

Log. of  $e$ , in seconds, 4.70415

Log.  $e \sin \epsilon$ , 4.45171 negative.

Thence  $e \sin \epsilon = -28295'' = 7^{\circ} 51' 35''$ .

Resuming the above logarithms, to get their differences, we have

Log.  $28295'' = 4.45171$

Log.  $28285'' = 4.45155$

Difference,  $10'' = 16$ . Therefore,  $1.6 =$  difference for one second, that is,  $= \mu$ . The formula  $\mu - \lambda = 1.28''$

By hypothesis  $nt = 332^{\circ} 28' 54.77''$

$e \sin \epsilon = -7^{\circ} 51' 35''$ . Therefore,

$nt + e \sin \epsilon = 324^{\circ} 37' 20''$ .

$\therefore nt + e \sin \epsilon = 324^{\circ} 37' 20'' = 4960$  seconds.

Hence,  $\frac{0.32''}{1.28''} \times 4960 = 1240$  seconds =  $20' 40''$ ; wherefore

the corrected value of  $\mathcal{U}$  is  $324^{\circ} 37' 20'' - 20' 40'' = 324^{\circ} 16' 40''$ .

To arrive at a greater accuracy, we repeat the same process with this result as a new assumed value, using the time tables of logs. to seven figures :

$$\text{Log. } e = 4.7041513$$

$$\text{Log. Sin } \varepsilon = 9.7663058$$

$$e \text{ Sin } \varepsilon, \text{ Log. } = 4.4704571 = -29543'' 18 = -8^{\circ} 23''$$

18. Therefore,  $nt + e \sin \varepsilon = 324^{\circ} 16' 31'' 59$ .

This differs from  $\varepsilon$  by  $8'' 41$ , and this, multiplied by  $\frac{\lambda}{\mu - \lambda} = \frac{29.25}{117.75}$ , produces  $2'' 09$ ; whence, finally, the corrected value of  $u$ , or the eccentric anomaly required, is  $324^{\circ} 16' 31'' 59 - 2'' 09 = 324^{\circ} 16' 29'' 5$ , exact within  $\frac{1}{100}$  of a second.

#### PROBLEM XVI.—*Tables of the Sun.*

Tables of the sun have been computed by astronomers, which give his mean longitude at the beginning of a certain year, as 1800, and at the beginning of the preceding and following years. There are supplementary tables, by means of which the use of the tables may be extended to 10,000 years before and after 1800. Tables of the sun also give the mean longitude of the perigee in the same way. In order to have the mean longitude at any instant, it is necessary to add to his mean longitude, at the beginning of the year, the mean daily motion in longitude, multiplied by the number of days and fractions of a day elapsed since the beginning of the year. This computation is facilitated by tables which give the motion of longitude from the beginning of each year to the first of each month; then tables for days, which give the sun's mean motion in longitude for any number of days, from one up to thirty-one; then tables for hours, giving the sun's mean motion in longitude, from one up to twenty-four; then for minutes, from one to sixty; and so for seconds.

*Example.*

To have the mean longitude of the sun, June 20th, 9h. 40m. 50s., 1853.

Take out from the tables of years the mean longitude for 1853; also, from the tables of months, the mean motion in longitude against June; from the tables of days, the mean motion in longitude against 20th; and so on through the hours, minutes, and seconds. Add up all these numbers, and the result will be the mean longitude of the sun at the instant required. The difference between the mean longitude of the sun and that of the perigee, at any time, is the mean anomaly at that instant. The tables of the sun give the equation of the center, corresponding to any given mean anomaly. (The tables are computed from formulæ such as we have had already.) The equation of the center, applied as a correction to the mean longitude, gives the true longitude of the sun for a perfectly elliptical orbit (the orbit would be a perfect ellipse if there were but the sun and earth), estimated from the mean equinox.

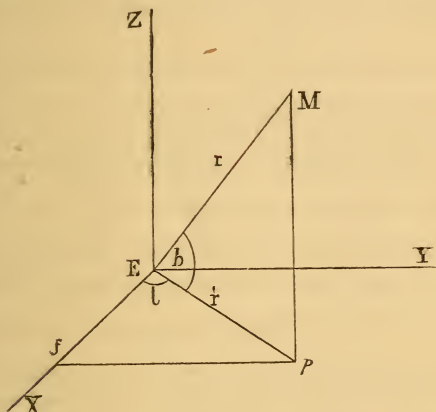
NOTE.—*To have the true longitude from the true equinox, the equation of the equinoxes must be applied, of which hereafter.*

Tables have also been computed for furnishing the effects of the perturbations by the different planets, in order that, being applied to the elliptical place of the sun, his true longitude for the given instant may be obtained. These last are tables of double entry, as the disturbing effects of a planet will depend upon the relative position of the three bodies—the earth, sun, and planet; the two arguments of the tables of perturbation are ephemeral. Finally, the value of the radius vector of the earth's orbit, computed on the supposition that the earth's orbit is a perfect ellipse, the semi-major axis of which is unity, is furnished by a table; and other tables give the variations of that element, produced by planetary perturbations. The value of radius vector for these tables may be computed from the polar equation of the ellipse, or from the equation  $r=a(1 - e \cos u)$ ,  $u$  being the eccentric anomaly.

PROBLEM XVII.—*Orbit of the Moon. Longitude of the node and inclination.*

From the observed right ascension and declination of the moon, the longitude and latitude may be computed. (Vid. *Trigonometry*, page 139.) If, from a series of such observations, a record of corresponding latitudes be made, and if, on inspection, the latitude at any time be found equal to zero, the moon is then in the plane of the ecliptic, or in one of the nodes of her orbit. The corresponding longitude of the moon, given by the record, is the longitude of the node. The maximum of all the recorded latitudes would evidently be the inclination of the orbit. As no observation may give a latitude exactly zero, the instant of its being zero may be found by proportion, by means of two latitudes, the one a little north, the other a little south of the ecliptic. The proportion would run as follows:—As the whole change of latitude in the interval between two observations (equal to the sum of the north and south latitude) is to the interval of time between the two, so is the north latitude to the interval of time between the instant of observation furnished by the north latitude, and the instant that the moon is in the node. Then another proportion:—As the whole interval of time between the observations is to the whole change of longitude in the interval, so is the interval found by the preceding proportion, to the difference of longitude of the moon at the first observation and the node. The same method will apply to a planet.

PROBLEM XVIII.—*Method of determining the inclination and longitude of the node by two observations of the moon.*



Let E be the centre of the earth and the origin of three co-ordinate axes, the plane of X Y being the plane of the ecliptic, and the axis of X the line of equinoxes, M the moon, E M the radius vector represented by  $r$ , E  $p$  projection of the radius vector represented by  $r'$ , the angle M E  $p$  or  $b$  will be the latitude, X E  $p$  or  $l$  her longitude. We have evidently the following equations, since M  $p = z$   $pf = y$  E  $f = X$ , the co-ordinates of the moon, viz.:

1.  $z = r \sin. b$
  2.  $y = r' \sin. l$
  3.  $x = r' \cos. l$
  4.  $r' = r \cos. b$
- Substituting in the second and third, for  $r'$ , its value obtained in the fourth.
- $$\left. \begin{array}{l} x = r \cos b \cos l \\ y = r \cos b \sin l \\ z = r \sin b \end{array} \right\} \quad \text{Q}$$

Three co-ordinates of the moon in space are thus expressed in terms of her geocentric latitude, longitude, and radius vector. The general equation of a plane passing through the origin of co-ordinates is  $Ax + By + Cz = 0$ .

Dividing throughout by A, it becomes  $x + \frac{B}{A}y + \frac{C}{A}z = 0$  ( $\omega$ )

Substituting for  $x$ ,  $y$ , and  $z$  in this, their values given by equation (Q) for two places of the moon, and dividing throughout by  $r$ , we have two equations and but two unknown quantities,  $\frac{B}{A}$  and  $\frac{C}{A}$ . Substituting in ( $\omega$ ) the values of

$\frac{B}{A}$  and  $\frac{C}{A}$  found from these two equations, we shall have the equation of the plane passing through the two positions of the moon and the center of the earth; that is, the plane of the moon's orbit. Making  $z = 0$  in equation ( $\omega$ ), and resolving it with respect to  $y$ , we have  $y = \frac{A}{B}x$  for the equation

of the trace on the plane of X Y. This trace, which is the line of nodes, makes an angle whose tangent is  $\frac{A}{B}$  with the axis of X or line of equinoxes. The value of A

and B having been already found, as shown above, the angle which the line of nodes makes with the line of equinoxes; in other words, the geocentric longitude of the node becomes known. As for the angle which a plane makes with the plane of X Y, we have (vid. *Analytical Geometry*)

$\cos. \text{ angle} = \frac{A}{\sqrt{1+B^2+C^2}}$ , and this is the inclination of the plane of the orbit with the plane of the ecliptic.

#### PROBLEM XIX.—*Retrogradation of the Moon's node.*

The longitude of the node will be found (as above) to be different at every revolution of the moon. The node moves back from east to west, and this is called the retrogradation of the moon's node. This retrogradation is found to amount to  $360^\circ$ , or, the line of nodes makes a complete revolution about the earth, so that the longitude of the node becomes the same again, estimated from the true equinox in 6798 days, 12 hours, 57 minutes, 52 seconds, or in about  $18\frac{3}{4}$  years. The physical cause of this phenomenon is the disturbing action of the sun. The sun's action, when the moon is out of the plane of the ecliptic, may be decomposed into

two, one acting in the plane of the ecliptic, the other perpendicular to it, which latter component has a tendency to draw the moon to this plane sooner than it would reach it if it had been undisturbed, and this occasions a precession or retrogradation of the nodes. The precession of the equinoxes may be explained in the same way, if we suppose a number of moons following each other in the same orbit, each of these will follow the above law, and this would be the case if the moons were contiguous, forming a ring. The same phenomenon would occur, though in a less degree, if the ring were diminished till it clasped the earth. Now, the earth is a spheroid, having its polar diameter shorter than its equatorial: it may, therefore, be regarded as a sphere clasped by a ring of matter, the thickness of which is greater at the equator, and diminishes to 0 at the poles.

#### PROBLEM XX.—*Nutation.*

The action upon the protuberant mass at the equator, producing the phenomenon of precession, is not the action of one body alone, but the combined action of the sun and moon, and must depend on the relative position of the sun, moon, and earth. This relative position must become the same after each revolution of the moon's nodes; there must, therefore, be a variation of the mean amount of precession, which goes through a period of  $18\frac{3}{4}$  years. This variation is called nutation, and the correction to be applied in consequence of it, given by the tables, is called the equation of equinoxes. The phenomenon of nutation was first discovered by Dr. Bradley. He found that the star Gamma Draconis varied its apparent place, or right ascension and declination, and that this variation completed its period in the time of revolution of the moon's node, or in  $18\frac{3}{4}$  years; and that the variation could be accounted for by supposing that the true pole of the heaven described a small ellipse about the mean pole, the semi-axes of which are  $9''$  and  $7''$ .

#### PROBLEM XXI.—*Aberration of light.*

This phenomenon was also discovered by Dr. Bradley, at the same place and with the same instruments. A va-

riation in the right ascension and declination of a star, as Gamma Draconis, was noticed to complete its period in a year, and could be accounted for by supposing that the apparent place of the star described a small ellipse about its true place, every year, whose major-axis is  $20''$ , and whose minor-axis is to the major as the sine of the stars' latitude is to unity. A phenomenon thus evidently connected with the motion of the earth in its orbit, and which may exactly be accounted for by supposing it to arise from the combined motion of the earth with that of light, (the velocity of light, previously discovered by means of the eclipses of Jupiter's satellites, to be such as to pass over the diameter of the earth's orbit into  $16''$  of time: by comparing the observations of the eclipses of the satellites when Jupiter was in opposition, with those when he was in conjunction, *i. e.* when the earth was on that part of its orbit nearest to Jupiter, and when on the opposite side of the orbit most remote from Jupiter.) The effect of these combined motions may be illustrated by rain falling vertically, which, to a person in motion, appears to fall obliquely towards him; so light, coming from a star to an observer in space with the earth in its orbit, will appear to come from a point a little forward of the true place of the star. A parallelogram of velocities being constructed, with its sides proportioned to the relative velocities of light and the earth, the diagonal of this parallelogram will make an angle, with its longest side, equal to the angular displacement of the star. As the earth is moving, in all possible directions in space, in the course of a year, the displacement of the star will be in all directions from its true place, and will, evidently, describe an oval curve about its true place in the heavens. If a star be in the pole of the ecliptic, its displacement, occasioned by the motion of the earth in the plane of the ecliptic, in a direction nearly perpendicular to the light from the star, will be, throughout the year, the same; and the star will appear to describe a circle about its true place equal to the maximum of aberration, or constant of aberration,  $= 20''$ . If the star be in the plane of the ecliptic, the parallelogram of velocities being, in this case, always in this plane, the angle of displacement will be always in this plane, and, consequently,

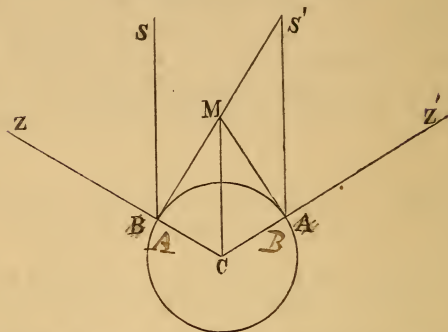
the ellipse of aberration degenerates into a straight line. If the star be anywhere between the plane of the ecliptic and its pole, the ellipse will be more or less eccentric. (Nutation or aberration, in right ascension and declination, or in latitude and longitude, *i. e.* the effect of these phenomena upon these co-ordinates, being very small, formulæ for their computation may be best obtained by aid of the "Differential Calculus." See Gauss' "Theoria Motus Corporum Cœlestium.")

PROBLEM XXII.—*Continuation of the determination of the elements of the Moon's orbit.*

By the solution of a right-angled spherical triangle, of which the inclination of the moon's orbit is one of the oblique angles, and the difference of longitude of the moon and the node of her orbit the base; or instead of the latter using the latitude of the moon, as the other perpendicular side of the triangle, the hypotenuse may be computed, which will be the angular distance of the moon from the node.

NOTE.—The retrogradation of the node need not be noticed in the computation, as it has no effect on the quantity obtained. If a series of such angular distance be computed from the observed right ascension and declination of the moon when on the meridian, and the interval of time between the observations exactly noted, the daily angular motion of the moon in her orbit may be obtained. The relative distances of the moon from the earth may be obtained approximately by means of her apparent diameter; or the absolute distances from her horizontal parallax. By means of the moon's daily angular motion in her orbit, and her daily variations of distance from the earth, the ellipse, which is her orbit and the position of perigee, may be obtained in a manner precisely analogous to that employed for the orbit of the earth, or apparent orbit of the sun.

PROBLEM XXIII.—*Method of determining the horizontal parallax of the Moon.*



Suppose that two observers in distant latitudes, and, for the sake of simplicity, on the same meridian, observe the difference of zenith distances, between the moon and some fixed star. Let A and B be the two places of observation, M the moon, AS and BS' parallels in direction of the star at an infinite distance. Then (*Trigonometry*, p. 275) since AMC and BMC are the parallaxes in altitude, we have the horizontal parallax  $H = AMC \div \sin MAZ$ . Therefore  $H \times \sin MAZ = AMC$ . Similarly, at station B,  $H \times \sin MBZ' = BMC$ : by addition,  $H (MAZ + MBZ') = AMC + BMC$ .

$$\therefore H = \frac{AMC + BMC}{\sin MAZ + \sin MBZ'} = \frac{AMB}{\text{last denominator.}} \text{ but}$$

$AMB = S' + MB S' = MAS + MB S'$ ; i.e. the sum of the differences of zenith distances of the moon and star. If the observers are not on the same meridian, allowance must be made for the change of the moon's declination in the interval of her transits over the two meridians (given by *Nautical Almanac*). The observed zenith distances should be reduced to geocentric zenith distances, by a method similar to that for converting astronomic to geocentric latitude. (*Trigonometry*, page 365.)

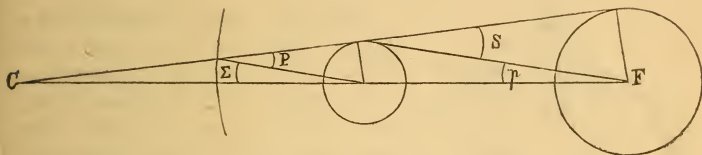
PROBLEM XXIV.—*Computation of the elements of a Lunar Eclipse.*

the other parts of the right-angled triangle may be computed. Let  $m n$  denote the moon's relative hourly motion in declination, *i. e.* the difference of hourly motion of the sun and moon, (given in Almanac,)  $M m =$  corresponding hourly motion of the moon on a parallel of declination  $M q$ , which is obtained by multiplying the difference of hourly motion of the sun and moon in right-ascension (given by Almanac) by  $\cos.$  of moon's declination.

By solution of the right-angled triangle  $M m n$ , which may be considered as a plane-triangle, the two perpendicular sides of which are thus given : we get  $M n$ , the moon's relative hourly motion in her orbit, and the angle  $m M n$ , which, subtracted from the angle  $S M q$ , before found by the solution of the triangle  $S M q$ , gives us the angle  $S M V$ , and consequently its supplement  $S M X$ , and then in the triangle  $S M X$ , having two sides and an angle, namely,  $S M$ , already computed, from triangle  $S M q$ , the side  $S X =$  the sum of the semi-diameters of the moon and section of shadow (the former of which is given by the Almanac, and the latter depends upon the parallaxes of the sun and moon, also given by the Almanac); we may compute  $M X$ ; the line,  $M X \div M n$  will give the hour and fractions to be applied to the instant the moon was at  $M$ , in order to have the instant of first contact of moon and shadow—completing the isosceles triangle  $S X Y$ , the center of the moon will evidently be at  $Y$  at the instant of last contact; then,  $\frac{M Y}{M n}$  will be the hours and fractions of an hour from the instant the moon is at  $M$ , to the instant of last contact, and  $M V \div M n$  will be the hours and fractions from the instant the moon is at  $M$ , to the instant of greatest phase.

PROBLEM XXV.—*Determination of the semi-diameter of the section of the shadow of the Earth at the distance of the Moon.*

DIAGRAM.



The interior angle  $C$  is equal to the exterior angle  $S$ , minus the other interior angle  $p$ , and the interior angle  $\Sigma$ , equals the exterior angle  $P$ , minus the interior angle  $C$ . Substituting the value of  $C$  above, we have  $\Sigma = P + p - S$ .

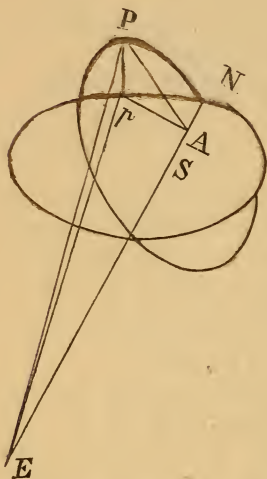
$\Sigma$  = semi-diameter of the shadow required.

$S$  = semi-diameter of the sun, *i. e.*, the visual angle at the earth, subtended by the radius of the sun.

$p$  = horizontal parallax of the sun or angle at the sun, subtended by the semi-diameter of the earth.  $P, p$  and  $S$  are given by the Nautical Almanac.  $P$  = horizontal parallax of the moon.

PROBLEM XXVI.—*Determination of the longitude of the node and the inclination of a planet's orbit.*

DIAGRAM.

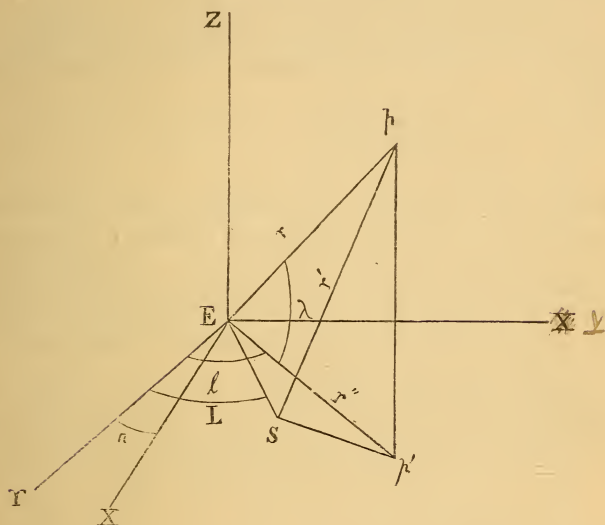


Let the upper ellipse be the planet's orbit, and the lower, the projection of the orbit on the plane of the ecliptic. S, the place of the sun; S N, the line of nodes. The longitude of N being determined by observations, (*i. e.*, by marking the time when the latitude is equal to zero,) wait till the geocentric longitude of the sun is equal to the longitude of the node; the earth will then be on the line of nodes. Suppose it to be at E; let P be the place of the planet at this time; let fall P *p* perpendicular to the plane of the ecliptic; draw *p* A perpendicular to E N,

and join P and A. The angle P A *p* is the inclination of the planet's orbit to the ecliptic (by definition). Then N E *p* is equal to the difference of longitude of the sun and the node of the planet, and the geocentric longitude of the planet; P E *p*, is equal to the geocentric latitude of the planet. In the triangles P E *p* and E A *p* right-angled at *p* and A, we have  $Pp = Ep \tan P E p$ .  $Ap = Ep \sin N E p$ .

Therefore, dividing the one by the other  $\frac{Pp}{Ap} = \tan P A p$   
 $= \frac{\tan P E p}{\sin N E p}$  or an expression for the value of the inclination of the planet's orbit in terms of the latitude and longitude. *Remark.*—The Heliocentric longitude of the node is the same, or differs from that of the earth when on the line of nodes, by 180 degrees.

PROBLEM XXVII.—*Determination of the elements of a planetary orbit.*



Let  $p$  be the place of a planet,  $p'$  its projection on the plane of  $X Y$ , for which, take the plane of the ecliptic and form the axis of  $X$ , a line through the center of the earth parallel to the line of nodes. The line  $E Y$  being the line of equinoxes, also in the plane of  $X Y$ . Denote the angle of these two lines or the heliocentric longitude of the node by  $n$ , the geocentric longitude of the sun by  $L$ , the geocentric longitude of the planet by  $l$ , its geocentric latitude by  $\lambda$ , the distance of the earth from the sun or radius vector of the terrestrial orbit by  $R$ , the distance of the planet from the earth by  $r$ , its projection by  $r''$ , from the sun or the radius vector of the planetary orbit, by  $r'$ ; the co-ordinates of the sun, by  $X. Y$ , of the planet, by  $x. y. z$ . Then, by drawing perpendiculars from  $S$  and  $p'$  to the axes of  $X$  and  $Y$ , we shall have right-angled plane triangles formed, which will give us the following :

$$X = R \cos (L - n), Y = R \sin (L - n) \dots \dots (1)$$

$$x = r'' \cos (l - n), y = r'' \sin (l - n) \dots \dots (2)$$

And from the right-angled triangle  $p E p'$

$$z = r \sin \lambda, r'' = r \cos \lambda \dots \dots (3)$$

Substituting in Equation (2) for  $r''$  its value in the second of equations (3) and writing after the results the first of equations (3)

$$\left. \begin{aligned} x &= r \cos \lambda \cos (l - n) \\ y &= r \cos \lambda \sin (l - n) \\ z &= r \sin \lambda \end{aligned} \right\} \quad (a)$$

Transfer now the origin of co-ordinates from E to S, *i. e.*, from the center of the earth to the center of the sun, and let the new axes be parallel to the old. Denoting the co-ordinates of the planet referred to, the new origin by  $x' y' z'$ , the formulas for transformation (see *Analytical Geometry*) are, since X and Y are the co-ordinates of the new origin,

$$x = X + x', y = Y + y', z = z'$$

$$\therefore x' = x - X, y' = y - Y, z' = z.$$

Substituting for X and Y their value, equation (1), and for  $x. y. z$ , their value in (a), these last become

$$\left. \begin{aligned} x' &= r \cos \lambda \cos (l - n) - R \cos (L - n) \\ y' &= r \cos \lambda \sin (l - n) - R \sin (L - n) \\ z' &= r \sin \lambda \end{aligned} \right\} \quad (b)$$

Squaring each of the group (b), and adding, observing that  $x'^2 + y'^2 + z'^2 = r'^2$ , we have  $r'^2 = r^2 \cos^2 \lambda [\cos^2 (l - n) + \sin^2 (l - n)] + R^2 [\cos^2 (L - n) + \sin^2 (L - n)] + r^2 \sin^2 \lambda - 2rR \cos \lambda [\cos (l - n) \cos (L - n) + \sin (l - n) \sin (L - n)]$ . But since  $\sin^2 + \cos^2$  of any arc = 1, the multipliers of  $R^2$  and  $r$ , equal to unity. The equation thus reduces to

$$r'^2 = r^2 (\cos^2 \lambda + \sin^2 \lambda) + R^2 - 2rR \cos \lambda [(l - n) - (L - n)]$$

$$\text{or } r'^2 = r^2 + R^2 - 2rR \cos \lambda \cos (l - L) \quad (c)$$

We may have another expression for the line  $pp' = z'$ . For if we conceive a perpendicular from  $p'$  to the line of nodes which runs through S parallel to the axis of X, this perpendicular will be equal to  ~~$z \sin \lambda$~~  and multiplied by the tang. of the inclination of the orbit, will give  $pp'$ , in symbols,  $i$  denoting the inclination

$$z' = (x' + X) \tan i.$$

$$(y - Y)$$

Substituting the values of  $z'$ ,  $x$ ,  $X$ , given by (a) (b) (1), this becomes

$$r \sin \lambda = r \cos \lambda \overset{\sin}{\cos} (l - n) - R \overset{\sin}{\cos} (L - n) \tan i.$$

This equation and equation (c) containing only two unknown quantities  $r$  and  $r'$ , these values may be found, and  $r$  being known,  $x'$ ,  $y'$  and  $z'$  the co-ordinates of the planet become known.

But  $x'$  is the projection of  $r'$  on the axis of  $X$ ; therefore,  $\frac{x'}{r'}$  is the value of the cosine of the angle which  $r'$  makes with this axis, i. e., the angle which the radius vector makes with the line of nodes, or the angular distance of the planet from the node.

The line  $r''$  being known from the geocentric longitude, and  $r$  by the last of equations (3). or by the equation  $r''^2 = r^2 - z'^2$ . Moreover,  $x'$  is the projection of  $r'$  on the axis,

therefore  $\frac{x'}{r'}$  is the cosine of the angle which the projection of  $r'$  makes with this axis, or the heliocentric longitude of the planet, estimated from the line of nodes.

**PROBLEM XXVIII.**—*Determination of the semi-major axis, eccentricity and longitude of the perihelion of a planet's orbit by means of three radii vectores and the corresponding angular distances of the planet from the node.*

Let  $r$ ,  $r'$ ,  $r''$  be the three radii vectores,

$v$ ,  $v'$ ,  $v''$  the angular distances from node.

$\pi$  the angular distance of the perihelion from the node.

$a$  the semi-major axis.

$e$  the eccentricity.

The polar equation of the ellipse (by *Analytical Geometry*) is

$$r = \frac{a(1 - e^2)}{1 + e \cos v} \quad (1)$$

in which  $v$  denotes the true anomaly or angular distance of

the planet from the perihelion, equal to the difference of the angular distances of the planet and perihelion from the node. Clearing eq. (1) of fractions, and substituting for  $v$  its value for each of the three observations, we have

$$1 + e \cos (\nu - \pi) = a (1 - e^2) \frac{1}{r} \quad (2)$$

$$1 + e \cos (\nu' - \pi) = a (1 - e^2) \frac{1}{r'} \quad (3)$$

$$1 + e \cos (\nu'' - \pi) = a (1 - e^2) \frac{1}{r''} \quad (4)$$

Subtracting (2) from (3) and (2) from (4)

$$e [\cos (\nu' - \pi) - \cos (\nu - \pi)] = a (1 - e^2) \left( \frac{1}{r'} - \frac{1}{r} \right) \quad (5)$$

$$e [\cos (\nu'' - \pi) - \cos (\nu - \pi)] = a (1 - e^2) \left( \frac{1}{r''} - \frac{1}{r} \right) \quad (6)$$

Dividing (5) by (6), and writing  $Q$  for the quotient of the second members, which will be a known quantity, since  $r$ ,  $r'$ ,  $r''$  are supposed to be known, we have

$$\frac{\cos (\nu' - \pi) - \cos (\nu - \pi)}{\cos (\nu'' - \pi) - \cos (\nu - \pi)} = Q.$$

or by *Trig.*, p. 101, (26)

$$\frac{\sin \frac{1}{2} (\nu' + \nu - 2 \pi) \sin \frac{1}{2} (\nu - \nu')}{\sin \frac{1}{2} (\nu'' + \nu - 2 \pi) \sin \frac{1}{2} (\nu - \nu'')} = Q.$$

Multiplying by the last factor of the denominator, and dividing by the last of the numerator, both known, the result in the second member will be known, and we have

$$\frac{\sin \frac{1}{2} (\nu' + \nu - 2 \pi)}{\sin \frac{1}{2} (\nu'' + \nu - 2 \pi)} = R.$$

Putting  $\sigma$  for  $\frac{1}{2} (\nu' + \nu)$  and  $\sigma'$  for  $\frac{1}{2} (\nu'' + \nu)$

$$\sin (\sigma - \pi) = R \sin (\sigma' - \pi)$$

or  $\sin \sigma \cos \pi - \cos \sigma \sin \pi = R (\sin \sigma' \cos \pi - \cos \sigma' \sin \pi)$  dividing throughout by  $\cos \pi$ .

$\sin \sigma - \cos \sigma \tan \pi = R (\sin \sigma' - \cos \sigma' \tan \pi)$ .  $\tan \pi$  being the only unknown in this last equation, its value may

be found, and being substituted in either two of eqs. (2) (3) (4), the values of  $a$  and  $e$  may be found from them.

The angular distance of the perihelion from the node, being known, by the solution of a right-angled spherical triangle, of which this angular distance is the hypotenuse, and the inclination of the orbit one of the oblique angles, the side adjacent this angle may be computed, and will be the longitude of the perihelion estimated from the node, by adding to which, the longitude of the latter, that of the perihelion will become known.





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